A NOTE ON THE ASSMUS–MATTSON THEOREM FOR SOME TERNARY CODES

EIICHI BANNAI, TSUYOSHI MIEZAKI, AND HIROYUKI NAKASORA*

Abstract. Let C be a two and three-weight ternary code. Furthermore, we assume that C_ℓ are t-designs for all ℓ by the Assmus–Mattson theorem. We show that $t \le 5$. As a corollary, we provide a new characterization of the (extended) ternary Golay code.

Key Words and Phrases. Assmus–Mattson theorem, t-designs, harmonic weight enumerator.

2010 *Mathematics Subject Classification.* Primary 05B05; Secondary 94B05, 20B25.

1. Introduction

In the present section, we explain our main results. Throughout this paper, C denotes a ternary $[n, k, d]$ code and we always assume that a combinatorial t-design allows the existence of repeated blocks. Let $C^\perp$ be a ternary $[n, n-k, d^\perp]$ dual code of C. We set $C_u := \{c \in C \mid \mathrm{wt}(c) = u\}$. We always assume that there exists $t \in \mathbb{N}$ that satisfies the following condition:

$$d^\perp - t = \sharp\{u \mid C_u \neq \emptyset, 0 < u \le n - t\}. \tag{1.1}$$

This is a condition of the Assmus–Mattson theorem (see Theorem 2.1), which we call the AM-condition. Let D_u and $D^\perp_w$ be the support designs of C and $C^\perp$ for weights u and w, respectively. Then, by (1.1) and Theorem 2.1, D_u and $D^\perp_w$ are t-designs (also s-designs for $0 < s < t$) for any u and w, respectively.

Let C satisfy the AM-condition. The main results of the present paper are the following theorems. For a two or three-weight code, we impose restrictions on $d^\perp$ and t.

Theorem 1.1. *Let C be a two-weight ternary code. If C satisfies the AM-condition, then one of the following holds:*

1. *$d^\perp = 5$ and C is the dual of the ternary Golay code $[11, 5, 6]$ with $t = 4$ or*
2. *$d^\perp \le 4$ and $t \le 3$.*

Theorem 1.2. *Let C be a three-weight ternary code. If C satisfies the AM-condition, then $d^\perp \le 6$ and $t \le 5$.*

*Corresponding author.

Theorem 1.3. *Let C be a three-weight ternary code, which has a weight n vector. If C satisfies the AM-condition, one of the following holds:*

(1) $d^{\perp} = 6$ *and C is the extended ternary Golay code* $[12, 6, 6]$ *with* $t = 5$ *or*
(2) $d^{\perp} \leq 5$ *and* $t \leq 4$.

It is interesting to note that Theorems 1.1 (1) and 1.3 (1) provide a new characterization of the (extended) ternary Golay code.

Let us explain the next result of the present paper. We introduce the following notations. Let D_w be the support design of a code C for weight w and

$$\delta(C) := \max\{t \in \mathbb{N} \mid \forall w, D_w \text{ is a } t\text{-design}\},$$
$$s(C) := \max\{t \in \mathbb{N} \mid \exists w \text{ s.t. } D_w \text{ is a } t\text{-design}\}.$$

We note that $\delta(C) \leq s(C)$. In our previous papers [7, 12, 19, 20, 21, 22, 23, 24], we considered the possible occurrence of $\delta(C) < s(C)$. This was motivated by Lehmer's conjecture, which is an analogue of $\delta(C) < s(C)$ in the theory of lattices and vertex operator algebras. For the details, see [5, 6, 8, 14, 16, 17, 19, 26, 27].

In [23], for an extremal Type III or IV code C', we prove the case $\delta(C') < s(C')$ does not occur. In [24], for a near-extremal Type I code C'' of length $n \equiv 0 \pmod 8$, we prove the case of $\delta(C'') < s(C'')$ occurs if and only if C'' is the unique near-extremal Type I $[16, 8, 4]$ code.

Therefore, in the present paper, we considered the possible occurrence of $\delta(C) < s(C)$. For cases in which $d^{\perp} - t = 1, 2$ or 3, the following theorem provides a criterion for n and d such that $\delta(C^{\perp}) < s(C^{\perp})$ occurs. Let $d = d_1$, and d_2 and d_3 be the second and third weights of C, respectively.

Theorem 1.4. *Let $\alpha_\ell = n - d_\ell - (t+1)$ and $\beta_\ell = d_\ell - (t+1)$ for $\ell = 1, 2$ or 3.*

(1) *Let C satisfy the AM-condition with $d^{\perp} - t = 1$. Let $w \in \mathbb{N}$ such that*

$$\sum_{i+j=w} 2^i \binom{\alpha_1}{i} \cdot (-1)^j \binom{\beta_1}{j} = 0.$$

Then $D^{\perp}_{w+t+1}$ is a $(t+1)$-design if $C^{\perp}_{w+t+1}$ is non-empty.

(2) *Let C satisfy the AM-condition with $d^{\perp} - t = 2$. Let $w \in \mathbb{N}$ such that*

$$\sum_{i+j=w} \left(2^i \binom{\alpha_1}{i} \cdot (-1)^j \binom{\beta_1}{j} - 2^i \binom{\alpha_2}{i} \cdot (-1)^j \binom{\beta_2}{j} \right) = 0.$$

Then $D^{\perp}_{w+t+1}$ is a $(t+1)$-design if $C^{\perp}_{w+t+1}$ is non-empty.

(3) *Let C satisfy the AM-condition with $d^{\perp} - t = 3$. Let $w \in \mathbb{N}$ such that*

$$\sum_{i+j=w} \left(2^i \binom{\alpha_1}{i} \cdot (-1)^j \binom{\beta_1}{j} - \frac{d_3 - d_1}{d_3 - d_2} 2^i \binom{\alpha_2}{i} \cdot (-1)^j \binom{\beta_2}{j} + \frac{d_2 - d_1}{d_3 - d_2} 2^i \binom{\alpha_3}{i} \cdot (-1)^j \binom{\beta_3}{j} \right) = 0.$$

Then $D^{\perp}_{w+t+1}$ is a $(t+1)$-design if $C^{\perp}_{w+t+1}$ is non-empty.

This theorem strengthens the Assmus–Mattson theorem for particular cases. We note that parameters n, d_i, and w that satisfy the condition in Theorem 1.4 are listed on the homepage of one of the authors [18]. In particular, we present the following corollary:

Corollary 1.5. *Let C satisfy the AM-condition in Theorem 1.4. For $n \leq 10$, in Miezaki's homepage* [18]*, we provide the parameters n, d_i, and w such that $\delta(C) < s(C)$ occurs.*

This paper is organized as follows: In Section 2, we provide background material and terminology. We review the concept of harmonic weight enumerators and theorems of designs, which we use to prove the main results. In Sections 3, 4, 5, and 6, we give proofs of Theorems 1.1, 1.2, 1.3, and 1.4, respectively. Finally, in Section 7, we conclude the paper with remarks.

We performed all the computer calculations in this paper with the help of Magma [9] and Mathematica [28].

2. Preliminaries

2.1. **Background material and terminology.** Let $\mathbb{F}_q$ be the finite field of q elements. A linear code C over $\mathbb{F}_q$ of length n is a subspace of $\mathbb{F}_q^n$. A ternary linear code C of length n is a subspace of $\mathbb{F}_3^n$. In the present paper, we always assume that C is a ternary code.

An inner product (x, y) on $\mathbb{F}_3^n$ is given by

$$(x, y) = \sum_{i=1}^{n} x_i y_i,$$

where $x, y \in \mathbb{F}_3^n$ with $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. The duality of a linear code C is defined as follows:

$$C^{\perp} = \{ y \in \mathbb{F}_3^n \mid (x, y) = 0 \text{ for all } x \in C \}.$$

A linear code C is self-dual if $C = C^{\perp}$. For $x \in \mathbb{F}_3^n$, the weight $\mathrm{wt}(x)$ is the number of its nonzero components. The minimum distance of code C is $\min\{\mathrm{wt}(x) \mid x \in C, x \neq 0\}$. A linear code of length n, dimension k, and minimum distance d is called an $[n, k, d]$ code (or $[n, k]$ code) and the dual code is called an $[n, n-k, d^{\perp}]$ code.

A t-(v, k, λ) design (or t-design, for short) is a pair $\mathcal{D} = (X, \mathcal{B})$, where X is a set of points of cardinality v, and $\mathcal{B}$ is a collection of k-element subsets of X called blocks, with the property that any t points are contained in precisely λ blocks.

The support of a nonzero vector $x := (x_1, \ldots, x_n)$, $x_i \in \mathbb{F}_3 = \{0, 1, 2\}$ is the set of indices of its nonzero coordinates: $\mathrm{supp}(x) = \{i \mid x_i \neq 0\}$. The support design of a code of length n for a given nonzero weight w is the design with points n of coordinate indices and blocks the supports of all codewords of weight w.

The following theorem is from Assmus and Mattson [1]. It is one of the most important theorems in coding theory and design theory:

Theorem 2.1 ([1])**.** *Let C be a linear code of length n over $\mathbb{F}_q$ with minimum weight d. Let $C^\perp$ denote the dual code of C, with minimum weight $d^\perp$. Suppose that an integer t $(1 \leq t \leq n)$ is such that there are at most $d - t$ weights of $C^\perp$ in $\{1, 2, \ldots, n - t\}$, or such that there are at most $d^\perp - t$ weights of C in $\{1, 2, \ldots, n - t\}$. Then the supports of the words of any fixed weight in C form a t-design (with possibly repeated blocks).*

2.2. **Harmonic weight enumerators.** In this subsection, we review the concept of harmonic weight enumerators.

Let C be a code of length n. The weight distribution of code C is the sequence $\{A_i \mid i = 0, 1, \ldots, n\}$, where A_i is the number of codewords of weight i. The polynomial

$$W_C(x, y) = \sum_{i=0}^{n} A_i x^{n-i} y^i$$

is called the weight enumerator of C. The weight enumerator of code C and its dual $C^\perp$ are related. The following theorem, proposed by MacWilliams, is called the MacWilliams identity:

Theorem 2.2 ([13])**.** *Let $W_C(x, y)$ be the weight enumerator of an $[n, k]$ code C over $\mathbb{F}_q$ and let $W_{C^\perp}(x, y)$ be the weight enumerator of the dual code $C^\perp$. Then*

$$W_{C^\perp}(x, y) = q^{-k} W_C(x + (q - 1)y, x - y).$$

A striking generalization of the MacWilliams identity was provided by Bachoc [2], who proposed the concept of harmonic weight enumerators. Harmonic weight enumerators have many applications; in particular, the relations between coding theory and design theory are reinterpreted and progressed by harmonic weight enumerators [2, 4]. For the reader's convenience, we quote the definitions and properties of discrete harmonic functions from [2, 10].

Let $\Omega = \{1, 2, \ldots, n\}$ be a finite set (which is the set of coordinates of the code) and let X be the set of its subsets, where for all $k = 0, 1, \ldots, n$, X_k is the set of its k-subsets. Let $\mathbb{R}X$ and $\mathbb{R}X_k$ denote the free real vector spaces spanned by the elements of X and X_k, respectively. An element of $\mathbb{R}X_k$ is denoted by

$$f = \sum_{z \in X_k} f(z) z$$

and identified with the real-valued function on X_k given by $z \mapsto f(z)$.

Such an element $f \in \mathbb{R}X_k$ can be extended to an element $\widetilde{f} \in \mathbb{R}X$ by setting, for all $u \in X$,

$$\widetilde{f}(u) = \sum_{z \in X_k, z \subset u} f(z).$$

If an element $g \in \mathbb{R}X$ is equal to some $\widetilde{f}$, for $f \in \mathbb{R}X_k$, we say that g has degree k. The differentiation γ is the operator defined by linearity from

$$\gamma(z) = \sum_{y \in X_{k-1}, y \subset z} y$$

for all $z \in X_k$ and for all $k = 0, 1, \ldots n$, and Harm_k is the kernel of γ:

$$\mathrm{Harm}_k = \ker(\gamma|_{\mathbb{R}X_k}).$$

Theorem 2.3 ([10, Theorem 7])**.** *A set of blocks $\mathcal{B} \subset X_m$, where $m \leq n$, is a t-design if and only if $\sum_{b \in \mathcal{B}} \widetilde{f}(b) = 0$ for all $f \in \mathrm{Harm}_k$, $1 \leq k \leq t$.*

In [2], the harmonic weight enumerator associated with a binary linear code C was defined as follows:

Definition 2.4. Let C be a binary code of length n and let $f \in \mathrm{Harm}_k$. The harmonic weight enumerator associated with C and f is

$$W_{C,f}(x, y) = \sum_{c \in C} \widetilde{f}(c) x^{n - \mathrm{wt}(c)} y^{\mathrm{wt}(c)}.$$

Bachoc and Tanabe proved the following MacWilliams-type equality:

Theorem 2.5 ([3, 25])**.** *Let $W_{C,f}(x, y)$ be the harmonic weight enumerator associated with the code C and the harmonic function f of degree k. Then*

$$W_{C,f}(x, y) = (xy)^k Z_{C,f}(x, y),$$

where $Z_{C,f}$ is a homogeneous polynomial of degree $n - 2k$, and satisfies

$$Z_{C^\perp,f}(x, y) = (-1)^k \frac{q^{n/2}}{|C|} Z_{C,f}\left(\frac{x + (q-1)y}{\sqrt{q}}, \frac{x - y}{\sqrt{q}}\right).$$

3. Proof of Theorem 1.1

Let C be a ternary code of length n. In this section, we always assume that C is a two-weight code and the weight distribution of C is "$0, d_1, d_2$". Before providing the proof, we present the following lemma:

Lemma 3.1. *Let $n, k \in \mathbb{Z}_{\geq 1}$. The solutions of the equation*

$$1 + 2n^2 = 3^k$$

are as follows:

$$(n, k) = (1, 1), (2, 2), (11, 5).$$

Proof. We assume that $k \equiv 0 \pmod 3$ and $x = 3^{k/3}$. Moreover, multiplying 2^3, we have

$$(2^2 n)^2 = (2 \cdot x)^3 - 2^3.$$

Let $Y = 2^2 n$ and $X = 2x$. Then

$$Y^2 = X^3 - 2^3.$$

By MAGMA,

```
E := EllipticCurve([ 0, -2^3 ]);
IntegralPoints(E);
```

we do not obtain any integer solutions (n, k).

Similarly, for the other cases $k \equiv 1, 2 \pmod 3$, we obtain the integer solutions $(1, 1), (2, 2), (11, 5)$. □

3.1. **Proof of Theorem 1.1.** In this subsection, we provide the proof of Theorem 1.1. Let

$$W_C(x, y) = x^n + \alpha x^{n-d_1} y^{d_1} + \beta x^{n-d_2} y^{d_2}$$

be the weight enumerator of C.

By Theorem 2.2,

$$\begin{aligned} W_{C^\perp}(x, y) =& 3^{-k} W_C(x + 2y, x - y) \\ =& 3^{-k} \sum_{i \geq 0} A_i x^{n-i} y^i. \end{aligned}$$

If $d^\perp \geq 5$, then $A_i = 0$ for $i \in \{1, \ldots, 4\}$; hence, we have the following constraints:

$$A_1 = 0, \tag{3.1}$$

$$A_2 = 0, \tag{3.2}$$

$$A_3 = 0, \tag{3.3}$$

$$A_4 = 0. \tag{3.4}$$

We note that the explicit values of A_i $(1 \leq i \leq 4)$ are listed in Section A.

Therefore, if $d^\perp \geq 5$, then we have the constraints in Eqs. (3.1)–(3.4). Using Eqs. (3.1)–(3.4), we present the following theorem:

Theorem 3.2. *If $d^\perp \geq 5$ then C is the dual of the ternary Golay code* $[11, 5, 6]$.

Proof. We assume that C has $d^\perp \geq 5$. Using Eqs. (3.1) and (3.2), we write α and β in terms of n, d_1, and d_2, that is,

$$\begin{aligned} \alpha = \alpha_1 = Y_{11}(n, d_1, d_2), \\ \beta = \beta_1 = Y_{12}(n, d_1, d_2). \end{aligned}$$

Similarly, using Eqs.(3.1) and (3.3), we write α and β in terms of n, d_1, and d_2, that is,

$$\alpha = \alpha_2 = Y_{21}(n, d_1, d_2),$$
$$\beta = \beta_2 = Y_{22}(n, d_1, d_2),$$

and using Eqs.(3.1) and (3.4), we write α and β in terms of n, d_1, and d_2, that is,

$$\alpha = \alpha_3 = Y_{31}(n, d_1, d_2),$$
$$\beta = \beta_3 = Y_{32}(n, d_1, d_2).$$

We note that the explicit values of Y_{ij} $(1 \le i \le 3, 1 \le j \le 2)$ are listed in Section B.

Using MATHEMATICA, we obtain the solutions of

$$\alpha_1 = \alpha_2, \alpha_1 = \alpha_3, \beta_1 = \beta_2, \beta_1 = \beta_3.$$

We note that these solutions are listed in Section C. The solutions (1)–(3), (6), and (7) are impossible. We show that if (4) and (5) occur then C is a code with $(n, k) = (11, 5)$.

Then using (4), Eqs. (3.1)–(3.4), and MATHEMATICA, we obtain

$$1 + 2n^2 = 3^k = |C|.$$

By Lemma 3.1,

$$(n, k) = (1, 1), (2, 2), (11, 5),$$

and it is clear that the first two cases are impossible and the last case occurs, which is the dual of the extended Golay code $[11, 5, 6]$. □

4. Proof of Theorem 1.2

Let C be a ternary code of length n. In this section, we always assume that C is a three-weight code and the weight distribution of C is "$0, d_1, d_2, d_3$". Before providing the proof, we present the following lemma:

Lemma 4.1. *Let $n, k \in \mathbb{Z}_{\ge 1}$. The solutions of the equation*

$$1 + \frac{8}{3}n - 2n^2 + \frac{4}{3}n^3 = 3^k$$

are as follows:

$$(n, k) = (1, 1), (2, 2), (3, 3).$$

Proof. We assume that $k \equiv 0 \pmod 2$ and $y = 3^{k/2}$. Then multiplying $3 \cdot 4^2$

$$(3 \cdot 4n)^3 - 6 \cdot 3(3 \cdot 4n)^2 + 8 \cdot 4 \cdot 3^2(3 \cdot 4n) + 3^4 \cdot 4^2 = (3^2 \cdot 4y)^2.$$

Let $Y = 3^2 \cdot 4y$ and $X = 3 \cdot 4n$. Then

$$X^3 - 6 \cdot 3X^2 + 8 \cdot 4 \cdot 3^2 X + 3^4 \cdot 4^2 = Y^2.$$

By MAGMA,

```
E := EllipticCurve ([0, -6*3, 0, 8*4*3^2, 3*4^2*3^3]);
IntegralPoints(E);
```

and we obtain the integer solutions (n, k):

$$(2, 2).$$

The other cases $k \equiv 1 \pmod 2$ can be proved similarly. □

4.1. **Proof of Theorem 1.2.** In this subsection, we provide the proof of Theorem 1.2. Let

$$W_C(x, y) = x^n + \alpha x^{n-d_1} y^{d_1} + \beta x^{n-d_2} y^{d_2} + \gamma x^{n-d_3} y^{d3}$$

be the weight enumerator of C.

By Theorem 2.2,

$$\begin{aligned} W_{C^\perp}(x, y) =& 3^{-k} W_C(x + 2y, x - y) \\ =& 3^{-k} \sum_{i \geq 0} A_i x^{n-i} y^i. \end{aligned}$$

If $d^\perp \geq 7$, then $A_i = 0$ for $i \in \{1, \dots, 5\}$; hence, we have the following constraints:

$$A_1 = 0, \tag{4.1}$$
$$A_2 = 0, \tag{4.2}$$
$$A_3 = 0, \tag{4.3}$$
$$A_4 = 0, \tag{4.4}$$
$$A_5 = 0, \tag{4.5}$$
$$A_6 = 0. \tag{4.6}$$

Using Eqs. (4.1)–(4.6), we present the following theorem:

Theorem 4.2. *There is no code C with $d^\perp \geq 7$.*

Proof. We assume that C has $d^\perp \geq 7$. Using Eqs. (4.1), (4.2), and (4.3), we write α, β, and γ in terms of n, d_1, and d_2, that is,

$$\begin{aligned} \alpha = \alpha_1 = Y_{11}(n, d_1, d_2), \\ \beta = \beta_1 = Y_{12}(n, d_1, d_2), \\ \gamma = \gamma_1 = Y_{13}(n, d_1, d_2). \end{aligned}$$

Similarly, using Eqs.(4.1), (4.2), and (4.4), we write α, β, and γ in terms of n, d_1, and d_2, that is,

$$\begin{aligned} \alpha = \alpha_2 = Y_{21}(n, d_1, d_2), \\ \beta = \beta_2 = Y_{22}(n, d_1, d_2), \\ \gamma = \gamma_2 = Y_{23}(n, d_1, d_2), \end{aligned}$$

using Eqs. (4.1), (4.2), and (4.5), we write α, β, and γ in terms of n, d_1, and d_2, that is,

$$\alpha = \alpha_3 = Y_{31}(n, d_1, d_2),$$
$$\beta = \beta_3 = Y_{32}(n, d_1, d_2),$$
$$\gamma = \gamma_3 = Y_{33}(n, d_1, d_2),$$

and using Eqs. (4.1), (4.2), and (4.6), we write α, β, and γ in terms of n, d_1, and d_2, that is,

$$\alpha = \alpha_4 = Y_{41}(n, d_1, d_2),$$
$$\beta = \beta_4 = Y_{42}(n, d_1, d_2),$$
$$\gamma = \gamma_4 = Y_{43}(n, d_1, d_2).$$

Using Mathematica, we obtain the solutions of

$$\alpha_1 = \alpha_2, \alpha_1 = \alpha_3, \alpha_1 = \alpha_4,$$
$$\beta_1 = \beta_2, \beta_1 = \beta_3, \beta_1 = \beta_4,$$
$$\gamma_1 = \gamma_2, \gamma_1 = \gamma_3, \gamma_1 = \gamma_4.$$

We note that these solutions are listed on the homepage of one of the authors [18]. The solutions (1)–(7) and (14)—(20) are impossible.

We assume that (8) occurs. The other cases can be proved similarly. Then using (8), Eqs. (4.1)–(4.6), and Mathematica, we obtain

$$1 + \frac{8}{3}n - 2n^2 + \frac{4}{3}n^3 = 3^k = |C|.$$

By Lemma 4.1,

$$(n, k) = (1, 1), (2, 2), (3, 3).$$

It is trivial that there is no code C with $(n, k) = (1, 1), (2, 2), (3, 3)$. □

5. Proof of Theorem 1.3

Let C be a ternary code of length n. In this section, we always assume that C is a three-weight code and the weight distribution of C is "$0, d_1, d_2, n$". Before providing the proof, we present the following lemma:

Lemma 5.1. *Let $n, k \in \mathbb{Z}_{\geq 1}$. The solutions of the equation*

$$9 - 12n + 6n^2 = 3^k$$

are as follows:

$$(n, k) = (1, 1), (2, 2), (3, 3), (12, 6).$$

Proof. We assume that $k \equiv 0 \pmod 3$ and $x = 3^{k/3}$. Let $n = N - 1$. Then $6N^2 = x^3 - 3$. Moreover, multiplying 6^3, we have

$$6^4N^2 = 6^3x^3 - 3 \cdot 6^3 \Leftrightarrow (6^2N)^2 = (6x)^3 - 3 \cdot 6^3.$$

Let $Y = 6^2N$ and $X = 6x$. Then

$$Y^2 = X^3 - 3 \cdot 6^3.$$

By MAGMA,

```
E := EllipticCurve([ 0, -3*6^3 ]);
IntegralPoints(E);
```

we obtain the integer solutions (n, k):

$$(3,3),\ (12,6).$$

The other cases $k \equiv 1, 2 \pmod 3$ can be proved similarly. ☐

5.1. **Proof of Theorem 1.3.** In this subsection, we provide the proof of Theorem 1.3. Let

$$W_C(x,y) = x^n + \alpha x^{n-d_1}y^{d_1} + \beta x^{n-d_2}y^{d_2} + \gamma y^n$$

be the weight enumerator of C.

By Theorem 2.2,

$$\begin{aligned} W_{C^\perp}(x,y) =& 3^{-k} W_C(x+2y, x-y) \\ =& 3^{-k} \sum_{i \geq 0} A_i x^{n-i} y^i. \end{aligned}$$

If $d^\perp \geq 6$, then $A_i = 0$ for $i \in \{1, \dots, 5\}$; hence, we have the following constraints:

$$A_1 = 0, \tag{5.1}$$

$$A_2 = 0, \tag{5.2}$$

$$A_3 = 0, \tag{5.3}$$

$$A_4 = 0, \tag{5.4}$$

$$A_5 = 0. \tag{5.5}$$

Therefore, if $d^\perp \geq 6$, then we have the constraints in Eqs. (5.1)–(5.5). Using Eqs. (5.1)–(5.5), we present the following theorem:

Theorem 5.2. *If $d^\perp \geq 6$, then C is the extended ternary Golay code* $[12, 6, 6]$.

Proof. We assume that C has $d^\perp \geq 6$. Using Eqs. (5.1), (5.2), and (5.3), we write α, β, and γ in terms of n, d_1, and d_2, that is,

$$\begin{aligned} \alpha = \alpha_1 = Y_{11}(n, d_1, d_2), \\ \beta = \beta_1 = Y_{12}(n, d_1, d_2), \\ \gamma = \gamma_1 = Y_{13}(n, d_1, d_2). \end{aligned}$$

Similarly, using Eqs.(5.1), (5.2), and (5.4), we write α, β, and γ in terms of n, d_1, and d_2, that is,

$$\begin{aligned} \alpha = \alpha_2 = Y_{21}(n, d_1, d_2), \\ \beta = \beta_2 = Y_{22}(n, d_1, d_2), \\ \gamma = \gamma_2 = Y_{23}(n, d_1, d_2), \end{aligned}$$

and using Eqs. (5.1), (5.2), and (5.5), we write α, β, and γ in terms of n, d_1, and d_2, that is,

$$\begin{aligned}\alpha = \alpha_3 = Y_{31}(n, d_1, d_2),\\ \beta = \beta_3 = Y_{32}(n, d_1, d_2),\\ \gamma = \gamma_3 = Y_{33}(n, d_1, d_2).\end{aligned}$$

Using Mathematica, we obtain the solutions of

$$\alpha_1 = \alpha_2, \alpha_1 = \alpha_3, \beta_1 = \beta_2, \beta_1 = \beta_3, \gamma_1 = \gamma_2, \gamma_1 = \gamma_3.$$

We note that these solutions are listed on the homepage of one of the authors [18]. The solutions (1)–(3), (6), and (7) are impossible. We show that if (4) and (5) occur then C is a code with $(n, k) = (12, 6)$.

Then using the (4), Eqs. (5.1)–(5.5), and Mathematica, we obtain

$$9 - 12n + 6n^2 = 3^k = |C|.$$

By lemma 5.1,

$$(n, k) = (1, 1), (2, 2), (3, 3), (12, 6),$$

and it is clear that the first three cases are impossible and the last case is the extended ternary Golay code $[12, 6, 6]$. □

6. Proof of Theorem 1.4

6.1. **Proof of Theorem 1.4 (1).** In this subsection, we provide the proof of Theorem 1.4 (1).

Proof. The harmonic weight enumerator of $f \in \mathrm{Harm}_{t+1}$ is

$$\begin{aligned}W_{C,f} =& ax^{n-d_1}y^{d_1}\\ =&(xy)^{t+1}ax^{n-d_1-(t+1)}y^{d_1-(t+1)},\end{aligned}$$

where $a \in \mathbb{R}$. We set

$$Z_{C,f} = ax^{n-d_1-(t+1)}y^{d_1-(t+1)}.$$

Then, by Theorem 2.5,

$$Z_{C^\perp,f} =a'(x+2y)^{n-d_1-(t+1)}(x-y)^{d_1-(t+1)}. \tag{6.1}$$

Let

$$W_{C^\perp,f} = (xy)^{t+1}Z_{C^\perp,f} = \sum p_i x^{n-i}y^i.$$

By (6.1),

$$\begin{aligned}p_{w+t+1} =&(\text{constant})\\ &\times \sum_{i+j=w} 2^i \binom{n-d_1-(t+1)}{i} \cdot (-1)^j \binom{d_1-(t+1)}{j}.\end{aligned}$$

By Theorem 2.3, if

$$\sum_{i+j=w} 2^i \binom{n-d_1-(t+1)}{i} \cdot (-1)^j \binom{d_1-(t+1)}{j} = 0,$$

then $D^{\perp}_{w+t+1}$ is a $(t+1)$-design.

□

6.2. **Proof of Theorem 1.4 (2).** In this subsection, we provide the proof of Theorem 1.4 (2).

Proof. The harmonic weight enumerator of $f \in \mathrm{Harm}_{t+1}$ is

$$\begin{aligned} W_{C,f} =& ax^{n-d_1}y^{d_1} + bx^{n-d_2}y^{d_2} \\ =& (xy)^{t+1}(ax^{n-d_1-(t+1)}y^{d_1-(t+1)} + bx^{n-d_2-(t+1)}y^{d_2-(t+1)}), \end{aligned}$$

where $a, b \in \mathbb{R}$. We set

$$Z_{C,f} = ax^{n-d_1-(t+1)}y^{d_1-(t+1)} + bx^{n-d_2-(t+1)}y^{d_2-(t+1)}.$$

Then, by Theorem 2.5,

$$\begin{aligned} Z_{C^{\perp},f} =& a'(x+2y)^{n-d_1-(t+1)}(x-y)^{d_1-(t+1)} \\ &+ b'(x+2y)^{n-d_2-(t+1)}(x-y)^{d_2-(t+1)}. \end{aligned}$$

Since $d^{\perp} \neq t+1$, the coefficients of $x^{n-2(t+1)}$ are zero. Then, $a'+b'=0$. Then,

$$\begin{aligned} Z_{C^{\perp},f} =& a'\big((x+2y)^{n-d_1-(t+1)}(x-y)^{d_1-(t+1)} \\ &- (x+2y)^{n-d_2-(t+1)}(x-y)^{d_2-(t+1)}\big). \end{aligned} \tag{6.2}$$

Let

$$W_{C^{\perp},f} = (xy)^{t+1}Z_{C^{\perp},f} = \sum p_i x^{n-i}y^i.$$

By (6.2),

$$\begin{aligned} p_{w+t+1} =& (\text{constant}) \\ &\times \sum_{i+j=w} \Bigg(2^i \binom{n-d_1-(t+1)}{i} \cdot (-1)^j \binom{d_1-(t+1)}{j} \\ &\qquad - 2^i \binom{n-d_2-(t+1)}{i} \cdot (-1)^j \binom{d_2-(t+1)}{j} \Bigg). \end{aligned}$$

By Theorem 2.3, if

$$\begin{aligned} &\sum_{i+j=w} \Bigg(2^i \binom{n-d_1-(t+1)}{i} \cdot (-1)^j \binom{d_1-(t+1)}{j} \\ &\qquad - 2^i \binom{n-d_2-(t+1)}{i} \cdot (-1)^j \binom{d_2-(t+1)}{j} \Bigg) = 0. \end{aligned}$$

Then $D^{\perp}_{w+t+1}$ is a $(t+1)$-design.

□

6.3. **Proof of Theorem 1.4 (3).** In this subsection, we provide the proof of Theorem 1.4 (3).

Proof. The harmonic weight enumerator of $f \in \mathrm{Harm}_{t+1}$ is

$$\begin{aligned} W_{C,f} =& a x^{n-d_1} y^{d_1} + b x^{n-d_2} y^{d_2} + c x^{n-d_3} y^{d_3} \\ =& (xy)^{t+1} (a x^{\alpha_1} y^{\beta_1} + b x^{\alpha_2} y^{\beta_2} + c x^{\alpha_3} y^{\beta_3}), \end{aligned}$$

where $a, b, c \in \mathbb{R}$, $\alpha_\ell = n - d_\ell - (t+1)$ and $\beta_\ell = d_\ell - (t+1)$ for $\ell = 1, 2$ or 3. We set

$$Z_{C,f} = a x^{\alpha_1} y^{\beta_1} + b x^{\alpha_2} y^{\beta_2} + c x^{\alpha_3} y^{\beta_3}.$$

Then, by Theorem 2.5,

$$\begin{aligned} Z_{C^\perp,f} =& a'(x+2y)^{\alpha_1}(x-y)^{\beta_1} \\ &+ b'(x+2y)^{\alpha_2}(x-y)^{\beta_2} \\ &+ c'(x+2y)^{\alpha_3}(x-y)^{\beta_3}. \end{aligned}$$

Since $d^\perp \neq t+1, t+2$, the coefficients of $x^{n-2(t+1)}$ and $x^{n-2(t+1)-1}y$ are zero. Then,

$$a' + b' + c' = 0, \tag{6.3}$$

$$a'(2\alpha_1 - \beta_1) + b'(2\alpha_2 - \beta_2) + c'(2\alpha_3 - \beta_3) = 0. \tag{6.4}$$

By (6.3) and (6.4),

$$\begin{aligned} b' &= -\frac{d_3 - d_1}{d_3 - d_2} a', \\ c' &= \frac{d_2 - d_1}{d_3 - d_2} a'. \end{aligned}$$

Then,

$$\begin{aligned} Z_{C^\perp,f} =& a'\big((x+2y)^{\alpha_1}(x-y)^{\beta_1} \\ &- \frac{d_3 - d_1}{d_3 - d_2}(x+2y)^{\alpha_2}(x-y)^{\beta_2} \\ &+ \frac{d_2 - d_1}{d_3 - d_2}(x+2y)^{\alpha_3}(x-y)^{\beta_3}\big). \end{aligned} \tag{6.5}$$

Let

$$W_{C^\perp,f} = (xy)^{t+1} Z_{C^\perp,f} = \sum p_i x^{n-i} y^i.$$

By (6.5),

$$p_{w+t+1} = (\text{constant}) \times \sum_{i+j=w} \left(2^i \binom{\alpha_1}{i} \cdot (-1)^j \binom{\beta_1}{j} - \frac{d_3 - d_1}{d_3 - d_2} 2^i \binom{\alpha_2}{i} \cdot (-1)^j \binom{\beta_2}{j} + \frac{d_2 - d_1}{d_3 - d_2} 2^i \binom{\alpha_3}{i} \cdot (-1)^j \binom{\beta_3}{j} \right).$$

By Theorem 2.3, if

$$\sum_{i+j=w} \left(2^i \binom{\alpha_1}{i} \cdot (-1)^j \binom{\beta_1}{j} - \frac{d_3 - d_1}{d_3 - d_2} 2^i \binom{\alpha_2}{i} \cdot (-1)^j \binom{\beta_2}{j} + \frac{d_2 - d_1}{d_3 - d_2} 2^i \binom{\alpha_3}{i} \cdot (-1)^j \binom{\beta_3}{j} \right) = 0.$$

Then $D_{w+t+1}^{\perp}$ is a $(t+1)$-design.

□

7. Concluding Remarks

Remark 7.1. (1) Are there examples that satisfy the condition of Theorem 1.4?

(2) For a two-weight code, if we assume that $d^{\perp} \geq 5$ and

$$W_C(x, y) = x^n + \alpha x^{n-d_1} y^{d_1} + \beta x^{n-d_2} y^{d_2},$$

then

$$1 + \alpha + \beta = \sum_{i=0}^{2} \binom{n}{i} 2^i.$$

Similarly, for a three-weight code, if we assume that $d^{\perp} \geq 7$ and

$$W_C(x, y) = x^n + \alpha x^{n-d_1} y^{d_1} + \beta x^{n-d_2} y^{d_2} + \gamma x^{n-d_3} y^{d_3},$$

then

$$1 + \alpha + \beta + \gamma = \sum_{i=0}^{3} \binom{n}{i} 2^i.$$

In the case $d_3 = n$, if we assume that $d^{\perp} \geq 6$, then

$$1 + \alpha + \beta + \gamma = 3 \sum_{i=0}^{2} \binom{n-1}{i} 2^i.$$

In [15], van Lint found the solutions of the following equation for $e = 2, 3$:

$$\sum_{i=0}^{e} \binom{n}{i} 2^i = 3^k.$$

Our method provides an alternative proof.

This suggests the following conjecture:

Conjecture 7.2. Let C be an ℓ-weight $[n.k.d]$ code over $\mathbb{F}_q$ and satisfy the AM-condition. If we assume that $d^{\perp} \geq 2\ell + 1$ and

$$W_C(x, y) = x^n + \sum_{1 \leq i \leq \ell} \alpha_i x^{n-d_i} y^{d_i},$$

then

$$1 + \alpha_1 + \alpha_2 + \cdots + \alpha_\ell = \sum_{i=0}^{\ell} \binom{n}{i} (q-1)^i = q^k.$$

Moreover, if $\ell \geq 4$, the codes corresponding the solutions of

$$\sum_{i=0}^{\ell} \binom{n}{i} (q-1)^i = q^k$$

do not exist. Hence, $d^{\perp} \leq 2\ell$ and $t \leq 2\ell - 1$ for $\ell \geq 4$.

To date, we do not have a proof of this conjecture.

Acknowledgments

The authors would like to thank the anonymous reviewers for their beneficial comments on an earlier version of the manuscript. The second named author was supported by JSPS KAKENHI (22K03277). We thank Edanz (https://jp.edanz.com/ac) for editing a draft of this manuscript.

Appendix A. Values of A_i

$A_1 = -3\alpha d_1 - 3\beta d_2 + 2n + 2\alpha n + 2\beta n$

$A_2 = 2(-1+n)n + 1/2\alpha(3d_1 + 9d_1^2 - 4n - 12d_1 n + 4n^2) + 1/2\beta(3d_2 + 9d_2^2 - 4n - 12d_2 n + 4n^2)$

$A_3 = 4/3(-2+n)(-1+n)n - 1/6\alpha(18d_1 + 27d_1^2 + 27d_1^3 - 16n - 54d_1 n - 54d_1^2 n + 24n^2 + 36d_1 n^2 - 8n^3) - 1/6\beta(18d_2 + 27d_2^2 + 27d_2^3 - 16n - 54d_2 n - 54d_2^2 n + 24n^2 + 36d_2 n^2 - 8n^3)$

$A_4 = 2/3(-3+n)(-2+n)(-1+n)n + \alpha(1/24(-3+d_1)(-2+d_1)(-1+d_1)d_1 - 1/3(-2+d_1)(-1+d_1)d_1(-d_1+n) + (-1+d_1)d_1(-1-d_1+n)(-d_1+n) - 4/3d_1(-2-d_1+n)(-1-d_1+n)(-d_1+n) + 2/3(-3-d_1+n)(-2-d_1+n)(-1-d_1+n)(-d_1+n)) + \beta(1/24(-3+d_2)(-2+d_2)(-1+d_2)d_2 - 1/3(-2+d_2)(-1+d_2)d_2(-d_2+n) + (-1+d_2)d_2(-1-d_2+n)(-d_2+n) - 4/3d_2(-2-d_2+n)(-1-d_2+n)(-d_2+n) + 2/3(-3-d_2+n)(-2-d_2+n)(-1-d_2+n)(-d_2+n))$

Appendix B. Values of Y_{ij}

$Y_{11} = -((2(-d_2 n + 3d_2^2 n - 2d_2 n^2))/(9d_1^2 d_2 - 9d_1 d_2^2 + 2d_1 n - 6d_1^2 n - 2d_2 n + 6d_2^2 n + 4d_1 n^2 - 4d_2 n^2))$

$Y_{12} = (2(d_1 n - 3d_1^2 n + 2d_1 n^2))/((-d_1 + d_2)(9d_1 d_2 + 2n - 6d_1 n - 6d_2 n + 4n^2))$

$Y_{21} = -((2(-2d_2 n + 9d_2^2 n + 9d_2^3 n - 6d_2 n^2 - 18d_2^2 n^2 + 8d_2 n^3))/((d_1 - d_2)(27d_1 d_2 + 27d_1^2 d_2 + 27d_1 d_2^2 + 4n - 18d_1 n - 18d_1^2 n - 18d_2 n - 72d_1 d_2 n - 18d_2^2 n + 12n^2 + 36d_1 n^2 + 36d_2 n^2 - 16n^3)))$

$Y_{22} = -((2(-2d_1n + 9d_1^2n + 9d_1^3n - 6d_1n^2 - 18d_1^2n^2 + 8d_1n^3))/((-d_1 + d_2)(27d_1d_2 + 27d_1^2d_2 + 27d_1d_2^2 + 4n - 18d_1n - 18d_1^2n - 18d_2n - 72d_1d_2n - 18d_2^2n + 12n^2 + 36d_1n^2 + 36d_2n^2 - 16n^3)))$

$Y_{31} = -((2(6d_2n - 27d_2^2n - 18d_2^3n - 9d_2^4n + 16d_2n^2 + 48d_2^2n^2 + 24d_2^3n^2 - 24d_2n^3 - 24d_2^2n^3 + 8d_2n^4))/((-d_1 + d_2)(81d_1d_2 + 54d_1^2d_2 + 27d_1^3d_2 + 54d_1d_2^2 + 27d_1^2d_2^2 + 27d_1d_2^3 + 12n - 54d_1n - 36d_1^2n - 18d_1^3n - 54d_2n - 180d_1d_2n - 90d_1^2d_2n - 36d_2^2n - 90d_1d_2^2n - 18d_2^3n + 32n^2 + 96d_1n^2 + 48d_1^2n^2 + 96d_2n^2 + 120d_1d_2n^2 + 48d_2^2n^2 - 48n^3 - 48d_1n^3 - 48d_2n^3 + 16n^4)))$

$Y_{32} = (2(6d_1n - 27d_1^2n - 18d_1^3n - 9d_1^4n + 16d_1n^2 + 48d_1^2n^2 + 24d_1^3n^2 - 24d_1n^3 - 24d_1^2n^3 + 8d_1n^4))/((-d_1 + d_2)(81d_1d_2 + 54d_1^2d_2 + 27d_1^3d_2 + 54d_1d_2^2 + 27d_1^2d_2^2 + 27d_1d_2^3 + 12n - 54d_1n - 36d_1^2n - 18d_1^3n - 54d_2n - 180d_1d_2n - 90d_1^2d_2n - 36d_2^2n - 90d_1d_2^2n - 18d_2^3n + 32n^2 + 96d_1n^2 + 48d_1^2n^2 + 96d_2n^2 + 120d_1d_2n^2 + 48d_2^2n^2 - 48n^3 - 48d_1n^3 - 48d_2n^3 + 16n^4))$

Appendix C. Solutions (1)–(7) in Proof of Theorem 3.2

$(1) d_1 = 0$

$(2) d_2 = 0$

$(3) n = 0$

$(4) d_1 = \frac{1}{6}\left(4n - \sqrt{8n-7} + 1\right), d_2 = \frac{-8n^2+12n+3\sqrt{8n-7}-1}{3\left(-4n+\sqrt{8n-7}+5\right)}$

$(5) d_1 = \frac{1}{6}\left(4n + \sqrt{8n-7} + 1\right), d_2 = \frac{8n^2-12n+3\sqrt{8n-7}+1}{3\left(4n+\sqrt{8n-7}-5\right)}$

$(6) d_1 = 1, n = 1$

$(7) d_2 = 1, n = 1$

References

[1] E.F. Assmus Jr. and H.F. Mattson Jr., New 5-designs, *J. Combin. Theory* **6** (1969), 122–151.

[2] C. Bachoc, On harmonic weight enumerators of binary codes, *Des. Codes Cryptogr.* **18** (1999), no. 1-3, 11–28.

[3] C. Bachoc, Harmonic weight enumerators of nonbinary codes and MacWilliams identities. Codes and association schemes (Piscataway, NJ, 1999), 1–23, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 56, Amer. Math. Soc., Providence, RI, 2001.

[4] E. Bannai, M. Koike, M. Shinohara, and M. Tagami, Spherical designs attached to extremal lattices and the modulo p property of Fourier coefficients of extremal modular forms, *Mosc. Math. J.* **6** (2006), 225–264.

[5] E. Bannai and T. Miezaki, Toy models for D. H. Lehmer's conjecture. *J. Math. Soc. Japan* **62** (2010), no. 3, 687–705.

[6] E. Bannai and T. Miezaki, Toy models for D. H. Lehmer's conjecture II. *Quadratic and higher degree forms*, 1–27, Dev. Math., **31**, Springer, New York, 2013.

[7] E. Bannai, T. Miezaki, and H. Nakasora, A note on the Assmus–Mattson theorem for some binary codes II, *Des. Codes Cryptogr.*, **91**, (2023), no. 7, 2509–2522.

[8] E. Bannai, T. Miezaki, and V.A. Yudin, An elementary approach to toy models for Lehmer's conjecture. (Russian) *Izv. Ross. Akad. Nauk Ser. Mat.* **75** (2011), no. 6, 3–16; *translation in Izv. Math.* **75** (2011), no. 6, 1093–1106.

[9] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, *J. Symbolic Comput.* **24** (1997), 235–265.

[10] P. Delsarte, Hahn polynomials, discrete harmonics, and t-designs, *SIAM J. Appl. Math.* **34** (1978), no. 1, 157–166.
[11] M. Harada and A. Munemasa, Database of self-dual codes: `https://www.math.is.tohoku.ac.jp/~munemasa/selfdualcodes.htm`
[12] N. Horiguchi, T. Miezaki, and H. Nakasora, On the support designs of extremal binary doubly even self-dual codes, *Des. Codes Cryptogr.*, **72** (2014), 529–537.
[13] J. Macwilliams, A theorem on the distribution of weights in a systematic code, *Bell System Tech. J.* **42** (1963), 79–84.
[14] D.H. Lehmer, The vanishing of Ramanujan's $\tau(n)$, *Duke Math. J.* **14** (1947), 429–433.
[15] J. H. van Lint, On the nonexistence of perfect 2-and 3-Hamming-error-correcting codes over $GF(q)$, *Inf. Control* **16** (1970), 396–401.
[16] T. Miezaki, Conformal designs and D.H. Lehmer's conjecture, *J. Algebra* **374** (2013), 59–65.
[17] T. Miezaki, Design-theoretic analogies between codes, lattices, and vertex operator algebras., *Des. Codes Cryptogr.* **89** (2021), no. 5, 763–780.
[18] T. Miezaki, Tsuyoshi Miezaki's website: `https://miezaki.w.waseda.jp/data.html`
[19] T. Miezaki, A. Munemasa, and H. Nakasora, A note on Assmus–Mattson type theorems, *Des. Codes Cryptogr.*, **89** (2021), 843–858.
[20] T. Miezaki and H. Nakasora, An upper bound of the value of t of the support t-designs of extremal binary doubly even self-dual codes, *Des. Codes Cryptogr.*, **79** (2016), 37–46.
[21] T. Miezaki and H. Nakasora, The support designs of the triply even binary codes of length 48, *J. Combin. Designs*, **27** (2019), 673–681.
[22] T. Miezaki and H. Nakasora, A note on the Assmus–Mattson theorem for some binary codes, *Des. Codes Cryptogr.*, **90** (2022), no. 6, 1485–1502.
[23] T. Miezaki and H. Nakasora, On the support t-designs of extremal Type III and IV codes, *Appl. Algebra Engrg. Comm. Comput.*, to appear.
[24] T. Miezaki and H. Nakasora, On the Assmus–Mattson type theorem for Type I and even formally self-dual codes, *J. Combin. Designs* **31** (2023), no. 7, 335–344.
[25] K. Tanabe, A new proof of the Assmus-Mattson theorem for non-binary codes, *Des. Codes Cryptogr.*, **22** (2001), no. 2, 149–155.
[26] B.B. Venkov, Even unimodular extremal lattices (Russian), *Algebraic geometry and its applications. Trudy Mat. Inst. Steklov.* **165** (1984), 43–48; translation in *Proc. Steklov Inst. Math.* **165** (1985) 47–52.
[27] B.B. Venkov, Réseaux et designs sphériques, (French) [Lattices and spherical designs] *Réseaux euclidiens, designs sphériques et formes modulaires*, 10–86, Monogr. Enseign. Math., 37, Enseignement Math., Geneva, 2001.
[28] Wolfram Research, Inc., Mathematica, Version 11.2, Champaign, IL (2017).

Faculty of Mathematics, Kyushu University (emeritus), Fukuoka, 819-0395 Japan
Email address: `bannai@math.kyushu-u.ac.jp`

Faculty of Science and Engineering, Waseda University, Tokyo 169–8555, Japan
Email address: `miezaki@waseda.jp`

Faculty of Computer Science and Systems Engineering, Okayama Prefectural University, Okayama, 719-1197 Japan
Email address: `nakasora@cse.oka-pu.ac.jp`